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Bianchi V Cosmology with a Specific Hubble Parameter

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Abstract In the present paper, we investigate the possibility of a variation law for Hubble's parameter H in the background of spatially homogeneous, anisotropic Bianchi type V space-time with perfect fluid source and time-dependent cosmological term. The model obtained presents a cosmological scenario which describes an early deceleration and late time acceleration. The model approaches isotropy and tends to a de Sitter universe at late times. The cosmological term Λ asymptotically tends to a genuine cosmological constant. It is observed that the solution is consistent with the results of recent observations.

Keywords Cosmological models · Hubble's parameter · Acceleration · Deceleration · Bianchi type models

1 Introduction

Recent astronomical observations of supernovae of type Ia suggest that the universe is undergoing an accelerated expansion with redshift $z \lesssim 1$ [1–4]. This has been confirmed by the observations of anisotropies in cosmic microwave background by the WMAP [5, 6]. This situation is not consistent with the present accepted physical theories. Assuming that gravity plays the main role in the structure and evolution of the universe and all usual types of matter with positive pressure generate atractive forces and decelerate expansion of the universe, one can not interpret the current accelerated expansion of our universe. In Einstein's theory of general relativity, to account for current accelerated phase of the universe, one needs to introduce a component to the matter fields of the universe with a large negative pressure, which is dubbed as dark energy. Astronomical observations indicate that our universe currently consists of approximately 70% dark energy, 25% dark matter and 5% baryonic matter and radiation. However, the absence of evidence on the nature of dark component gave origin to an intense debate, mainly, to theoretical speculations. Among a number of possibilities, to describe this dark energy component, the simplest and most-theoretically appealing

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way is by means of cosmological constant Λ. This exotic vacuum energy acts on Einstein's field equations as an isotropic and homogeneous source represented by a barotropic equation of state $p_\Lambda = -\rho_\Lambda$, where p_Λ and ρ_Λ are pessure and energy density of dark energy respectively. However, the dark energy needs to have its initial conditions properly 'tuned' in order to dominate the expansion of the universe at precisely the later stages of matter era so that nucleosynthesis in the early universe and large scale structure formation in the matter dominated regime could proceed smoothly. The observationally small current value of $\rho_\Lambda \sim 10^{-124}$ GeV^4 falls below the value resulting from quantum field theories by 120 orders of magnitude. To accommodate and to interpret the above facts, physical models with varying cosmological constant, especially time-dependent, have been proposed by a number of researchers [7–17]. In these models cosmological term Λ relaxes to its present small value through the expansion of the universe.

On the other hand, Hubble parameter H and deceleration parameter q are important observational quantities for any physically relevant model. The present day value H_0 of Hubble parameter sets the present time scale of the expansion while q_0, the present day value of deceleration parameter tells that expansion of the present universe is accelerating rather than going to decelerate as expected before the supernovae of type Ia observations [1–4]. However a number of recent observational data suggest that this accelerating phase of the universe is a recent phenomena. So, it is natural to assume that dark energy was insignificant in early evolution of the universe while it has the dominant contribution at the present accelerating epoch. The above facts indicate that a physically relevant model should have decelerating expansion in the early phase of matter era to allow the formation of large structures followed by late time accelerated phase during its expansion history. The transition from a decelerated phase to accelerated stage of evolution can be due to the dominance of dark energy over other kinds of matter fields in the later epoch. For this purpose, we need to have a deceleration parameter q, which is positive in the early epoch and become negative at late times. In a recent paper [18], we have proposed a form of Hubble parameter H as a function of the scale factor R in the background of spatially homogeneous and anisotropic Bianchi type I space-time in such a way that the resulting deceleration parameter q has the desired property of a signature flip. The variation of Hubble's parameter as assumed is consistent with the desired behavior of the universe. With this specific form of Hubble parameter, scale factors and other physical parameters are explicitly determined and we obtain a new class of cosmological models which leads to a cosmological scenario as an initial phase with decelerating expansion followed by an accelerated one at late times.

Bianchi type I space-times are the simplest anisotropic models which are spatially homogeneous with flat space slices but direction dependent rates of expansion or contraction. Bianchi type V models having richer structure than Bianchi type I space times are interesting to study. Moreover, Bianchi type V models are sufficiently more complicated than the simplest Bianchi type models and at the same time they are simple generalization of negative curvature FRW models. These models are favoured by the available evidences for low density universes. Roy and Singh [19, 20] have investigated Bianchi type V models with electromagnetic field. Banerjee and Sanyal [21] have considered Bianchi V cosmologies with viscosity and heat flow. Farnsworth [22], Beesham [23], Collins [24], Maartens and Nel [25], Coley [26], Roy and Prasad [27], Bali and Jain [28], Bali and Meena [29], Bali and Singh [30], Pradhan and Yadav [31], and Singh [32] have investigated Bianchi type V models in different physical contexts.

In this paper, we intend to extend the result of Singh [18] to Bianchi type V space-time with perfect fluid source. Exact solutions of Einstein's field equations are obtained and general features of the model are discussed.

2 Metric and Field Equations

Spatially homogeneous and anisotropic Bianchi type V space-time is given by the line element

$$ds^2 = -dt^2 + A^2(t)dx^2 + e^{2\alpha x}\left\{B^2(t)dy^2 + C^2(t)dz^2\right\}, \tag{1}$$

where α is a constant. We consider the cosmic matter to be perfect fluid represented by energy-momentum tensor

$$T_{ij} = (\rho + p)v_i v_j + p g_{ij}, \tag{2}$$

satisfying barotropic equation of state

$$p = \omega\rho \quad (0 \le \omega \le 1). \tag{3}$$

Here ρ is the energy density of the perfect fluid, p is the corresponding isotropic pressure and v^i, the flow vector of fluid satisfying $v_i v^i = -1$. In comoving system of coordinates, from (2), one finds

$$T_1^1 = T_2^2 = T_3^3 = -p \quad \text{and} \quad T_4^4 = \rho. \tag{4}$$

The Einstein field equations (in gravitational units $8\pi G = c = 1$) with variable cosmological term $\Lambda(t)$ are

$$R_i^j - \frac{1}{2}Rg_i^j = -T_i^j + \Lambda g_i^j. \tag{5}$$

For the Bianchi type V space-time with perfect fluid distribution, Einstein's fields equations (5) yield the following equations

$$p - \Lambda = \frac{\alpha^2}{A^2} - \frac{\ddot{B}}{B} - \frac{\ddot{C}}{C} - \frac{\dot{B}\dot{C}}{BC}, \tag{6}$$

$$p - \Lambda = \frac{\alpha^2}{A^2} - \frac{\ddot{A}}{A} - \frac{\ddot{C}}{C} - \frac{\dot{A}\dot{C}}{AC}, \tag{7}$$

$$p - \Lambda = \frac{\alpha^2}{A^2} - \frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} - \frac{\dot{A}\dot{B}}{AB}, \tag{8}$$

$$\rho + \Lambda = -\frac{3\alpha^2}{A^2} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{A}\dot{C}}{AC}, \tag{9}$$

$$0 = \frac{2\dot{A}}{A} - \frac{\dot{B}}{B} - \frac{\dot{C}}{C}. \tag{10}$$

Vanishing divergence of Einstein tensor $R_i^j - \frac{1}{2}Rg_i^j$ gives rise to

$$\dot{\rho} + (\rho + p)\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) + \dot{\Lambda} = 0. \tag{11}$$

In the above and elsewhere overhead dot $(\cdot)$ indicates ordinary time derivative of the concerned quantity.

From (11), we observe that in case of constant Λ, we recover the continuity equation of matter. In order to satisfy energy conservation, a decaying cosmological term Λ necessarily leads to matter creation [33].

We define the average scale factor R of Bianchi V space-time as $R^3 = ABC$. From (6), (7), (8) and (10), we obtain

$$\frac{\dot{A}}{A} = \frac{\dot{R}}{R}, \tag{12}$$

$$\frac{\dot{B}}{B} = \frac{\dot{R}}{R} - \frac{k}{R^3}, \tag{13}$$

$$\frac{\dot{C}}{C} = \frac{\dot{R}}{R} + \frac{k}{R^3}, \tag{14}$$

where k is constant of integration.

Equations (12)–(14), on integration give

$$A = m_1 R, \tag{15}$$

$$B = m_2 R \exp\left(-k \int \frac{dt}{R^3}\right), \tag{16}$$

$$C = m_3 R \exp\left(k \int \frac{dt}{R^3}\right), \tag{17}$$

where m_1, m_2 and m_3 are constants of integration satisfying $m_1 m_2 m_3 = 1$. Using suitable coordinate transformations, constants m_2 and m_3 can be absorbed. Therefore, we assume $m_2 = m_3 = 1$ implying $m_1 = 1$. We introduce volume expansion θ and shear scalar σ as usual

$$\theta = v^i{}_{;i} \quad \text{and} \quad \sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij}, \tag{18}$$

where σ_{ij} is the shear tensor and semicolon (;) stands for covariant differentiation. For the Bianchi type V model, expansion scalar θ and shear scalar σ are given by

$$\theta = \frac{3\dot{R}}{R}, \tag{19}$$

$$\sigma = \frac{k}{R^3}. \tag{20}$$

In analogy with FRW universe, we define a generalized Hubble parameter H and generalized deceleration parameter q as

$$H = \frac{\dot{R}}{R}, \tag{21}$$

$$q = -\frac{\ddot{R}}{RH^2}. \tag{22}$$

Equations (6)–(9) and (11) can be expressed in terms of H, σ and q as

$$p - \Lambda = \frac{\alpha^2}{R^2} + H^2(2q-1) - \sigma^2, \tag{23}$$

$$\rho + \Lambda = -\frac{3\alpha^2}{R^2} + 3H^2 - \sigma^2, \tag{24}$$

$$\dot{\rho} + 3(\rho + p)\frac{\dot{R}}{R} + \dot{\Lambda} = 0. \tag{25}$$

From (24), we observe that in Bianchi type V space-time energy density $\rho + \Lambda$ is smaller than the corresponding energy density in Bianchi type I space-time indicating that Bianchi type V model is a low density universe. Also from (24), we get

$$\frac{\sigma^2}{\theta^2} = \frac{1}{3} - \frac{\rho}{\theta^2} - \frac{3\alpha^2}{R^2\theta^2} - \frac{\Lambda}{\theta^2}. \tag{26}$$

Therefore, $0 < \frac{\sigma^2}{\theta^2} < \frac{1}{3}$ and $0 < \frac{\rho}{\theta^2} < \frac{1}{3}$ for $\Lambda \geq 0$. Thus a positive Λ puts restriction on the upper limit of anisotropy whereas a negative Λ gives more room for the anisotropy. From (23) and (24), we obtain

$$\frac{d\theta}{dt} = -\frac{1}{2}(\rho + 3p) - 2\sigma^2 - \frac{\theta^2}{3} + \Lambda, \tag{27}$$

which is Raychaudhuri equation for the given distribution. We observe that a positive Λ will slow down the rate of decrease. Also from (20), we obtain

$$\dot{\sigma} = -3\sigma H. \tag{28}$$

Thus, the energy density associated with the anisotropy σ^2 decays rapidly due to expansion and it becomes negligible for infinitely large value of R. From (23) and (24), we obtain

$$\frac{\ddot{R}}{R} = -\frac{1}{6}(\rho + 3p) - \frac{2\sigma^2}{3} + \frac{\Lambda}{3}. \tag{29}$$

We observe that active gravitational mass density and anisotropy contributes in driving the acceleration of the universe and a positive Λ arrests this acceleration. If we restrict ourselves to dust distribution only, (23) and (24) give

$$\frac{2\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k^2}{R^6} - \frac{\alpha^2}{R^2} - \Lambda = 0, \tag{30}$$

$$\frac{3\dot{R}^2}{R^2} = \frac{k^2}{R^6} + \frac{3\alpha^2}{R^2} + \rho + \Lambda. \tag{31}$$

We observe that for $\Lambda \geq 0$, each term on the right hand side of (31) is non-negative. Thus, $\dot{R}$ does not change sign and we get ever-expanding models. For $\Lambda < 0$, however, we can get universes that expand and then recontract.

3 Universe with Specific Hubble Parameter

In order to get an expanding model of the universe consistent with observations, one needs a Hubble parameter H such that the model starts with a decelerating expansion followed by an accelerated expansion at late times. Following a recent paper [18], we investigate the variation of Hubble parameter H as a one-parameter function

$$H(R) = a(R^{-n} + 1), \tag{32}$$

where $a(> 0)$ and $n(> 1)$ are constants, in the background of Bianchi type V space-time. For this choice, the deceleration parameter q comes out to be

$$q = \frac{n}{R^n + 1} - 1. \tag{33}$$

A similar form for q has also been proposed by Banerjee and Das [34] in case of Robertson-Walker space-time. From (33), we observe that when $R = 0$, $q = n - 1 > 0$, $q = 0$ for $R^n = n - 1$, and for $R^n > n - 1$, $q < 0$. We assume that $R = 0$ for $t = 0$. Therefore, the universe begins with a decelerating expansion and the expansion changes from decelerating phase to accelerating one. This cosmological scenario is in agreement with SNe Ia astronomical observations [1–4] and it presents a unified picture of the expansion history of the evolving universe. With the help of (20), (32) and (33), we can write (23) and (24) in terms of R as

$$p = \frac{\alpha^2}{R^2} - \frac{k^2}{3R^6} + a^2(R^{-n} + 1)^2\left(\frac{2n}{R^n + 1} - 3\right) + \Lambda, \tag{34}$$

$$\rho = -\frac{3\alpha^2}{R^2} - \frac{k^2}{3R^6} + 3a^2(R^{-n} + 1)^2 - \Lambda. \tag{35}$$

We observe that the model has singularity at $t = 0$ (i.e. $R = 0$). For large values of t (i.e. $R \to \infty$), we get $\rho = -p = 3a^2 - \Lambda$ implying that the universe is dominated by vacuum energy at late times. From (32) and (33) we obtain

$$H^{-1} = \frac{n - q - 1}{na}, \tag{36}$$

which indicates that H^{-1} increases as q decreases being maximum $(= a^{-1})$ for $q = -1$.

Integrating (32), we obtain

$$R^n = e^{na(t+t_1)} - 1, \tag{37}$$

where t_1 is constant of integration. Assuming $R = 0$ for $t = 0$, we get $t_1 = 0$. Therefore

$$R^n = e^{nat} - 1. \tag{38}$$

For the model, matter density ρ and cosmological term Λ are given by

$$\rho = \frac{2}{(\omega + 1)}\left\{\frac{na^2 e^{nat}}{(e^{nat} - 1)^2} - \frac{k^2}{(e^{nat} - 1)^{6/n}} - \frac{\alpha^2}{(e^{nat} - 1)^{2/n}}\right\}, \tag{39}$$

$$\Lambda = \frac{a^2 e^{nat}}{(e^{nat}-1)^2}\left(3e^{nat} - \frac{2n}{\omega+1}\right) + \frac{(1-\omega)k^2}{(\omega+1)(e^{nat}-1)^{6/n}} - \frac{(3\omega+1)\alpha^2}{(\omega+1)(e^{nat}-1)^{2/n}}. \tag{40}$$

Expansion scalar θ, shear σ and deceleration parameter q for the model are:

$$\theta = \frac{3a}{1-e^{-nat}}, \tag{41}$$

$$\sigma = \frac{k}{(e^{nat}-1)^{3/n}}, \tag{42}$$

$$q = \frac{n}{e^{nat}} - 1. \tag{43}$$

4 Discussion

We observe that spatial volume of the model universe is zero at $t = 0$. Therefore the model starts evolving at $t = 0$ and expands with cosmic time t. Matter density ρ, volume expansion θ and shear scalar σ diverge at $t = 0$. The model has singularity at $t = 0$. At $t = 0$, deceleration parameter $q = n - 1$ (> 0). Therefore the expansion in the model decelerates. In the limit of large times i.e. $t \to \infty$, matter density ρ and shear σ become negligible. Cosmological term Λ is infinite at $t = 0$. It asymptotically tends to a genuine cosmological constant and our solution tends to a de Sitter universe with $H = \sqrt{\Lambda/3} = a$. In the limit $t \to \infty$, we obtain $q = -1$ which characterizes the de Sitter solution [35]. Therefore, the model starts with a decelerating expansion and expansion in the model changes from the decelerating phase to an accelerating one. Time t_q when the expansion changes from the decelerating to accelerating phase can be obtained by equating $q = 0$ in (43) and it comes out to be

$$t_q = \frac{\ln n}{na}. \tag{44}$$

For the model,

$$\frac{\sigma}{\theta} = \frac{k(1-e^{-nat})}{3a(e^{nat}-1)^{3/n}}. \tag{45}$$

For large values of t, $\frac{\sigma}{\theta}$ tends to zero. Therefore, the model approaches isotropy at late times. The age of the universe is given by

$$nat_0 = \ln\left(\frac{H_0}{H_0 - a}\right), \tag{46}$$

where subscript ‘0’ indicates the present value of the concerned quantity. For $a \ll 1$ and $1 < n \leq 3$, variation of deceleration parameter q with cosmic time t is shown in Fig. 1.

5 Conclusion

The paper discusses the possibility of a variation law for Hubble parameter H as a function of average scale factor R in the background of homogeneous Bianchi type V space-time that

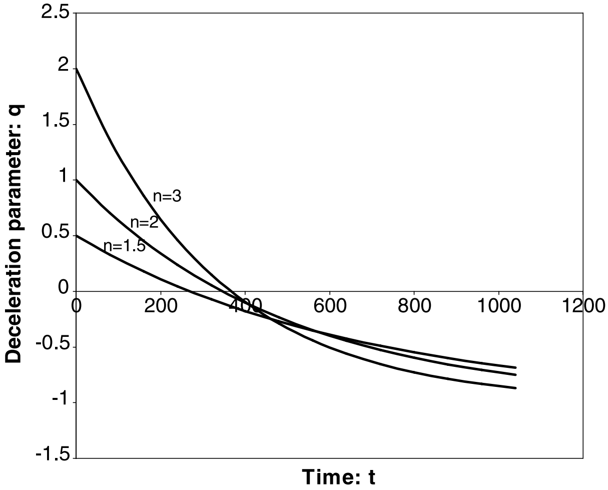

Fig. 1 Plot of deceleration parameter q vs time t with parameters $a = 0.001$ and $n = 1.5, 2, 3$

yields a deceleration parameter $q = q(R)$ having the property of signature flip. Exact solutions of Einstein's field equations have been obtained by using the variation law for H. The resulting model evolves with decelerating expansion in the initial epoch followed by a late time accelerated expansion. It presents a unified picture of the expansion history of universe in accordance with well-based features of modern cosmology. Observations of 16 type Ia supernovae, made by the Hubble Space Telescope (HST) [36], modified earlier astronomical results and provided conclusive evidence for deceleration prior to cosmic acceleration caused by dark energy in the recent past.

We observe that the model approaches isotropy for large values of t and cosmological term Λ, being very large at initial times, tends asymptotically to a genuine cosmological constant supported by the results from recent supernovae Ia observations. For large values of t, the model results in a de Sitter universe.

It is to mention here that the assumption of Hubble parameter H as a function of R is ad hoc in the sense that it does not result from a known field theory. However, it does produce a solution that presents appropriate description of the universe consistent with observations. The model definitely has problems, particularly that of fine tuning. The agreement with the observed universe is just qualitative. Precise observational tests are required to verify or disprove the model.

References

1. Riess, A.G., et al.: Astron. J. **116**, 1009 (1998)
2. Riess, A.G., et al.: Astrophys. J. **607**, 665 (2005)
3. Perlmutter, S., et al.: Astrophys. J. **517**, 565 (1999)
4. Astier, P., et al.: Astron. Astrophys. **447**, 31 (2006)
5. Spergel, D.N., et al.: Astrophys. J. Suppl. **148**, 175 (2003)
6. Spergel, D.N., et al.: Astrophys. J. Suppl. **170**, 377 (2007)
7. Ozer, M., Taha, M.O.: Nucl. Phys. B **287**, 776 (1987)
8. Freese, K., Adams, F.C., Frieman, J.A., Mottola, E.: Nucl. Phys. B **287**, 797 (1987)
9. Chen, W., Wu, Y.S.: Phys. Rev. D **41**, 695 (1990)
10. Carvalho, J.C., Lima, J.A.S., Waga, I.: Phys. Rev. D **46**, 2404 (1992)
11. Waga, I.: Astrophys. J. **414**, 436 (1993)
12. Saha, B.: Astrophys. Space Sci. **302**, 83 (2006)

13. Saha, B.: Int. J. Theor. Phys. **45**, 983 (2006)
14. Singh, C.P.: Int. J. Theor. Phys. **45**, 531 (2006)
15. Pradhan, A., Singh, S.K.: Int. J. Mod. Phys. D **13**, 503 (2004)
16. Vishwakarma, R.G.: Gen. Relativ. Gravit. **37**, 1305 (2005)
17. Singh, J.P., Tiwari, R.K.: Int. J. Mod. Phys. D **16**, 745 (2007)
18. Singh, J.P.: Astrophys. Space Sci. **318**, 103 (2008). DOI:10.1007/s10509-008-9890-8
19. Roy, S.R., Singh, J.P.: Astrophys. Space Sci. **96**, 303 (1983)
20. Roy, S.R., Singh, J.P.: Aust. J. Phys. **38**, 763 (1985)
21. Banerjee, A., Sanyal, A.K.: Gen. Relativ. Gravit. **20**, 103 (1988)
22. Farnsworth, D.L.: J. Math. Phys. **8**, 2315 (1967)
23. Beesham, A.: Astrophys. Space Sci. **125**, 99 (1986)
24. Collins, C.B.: Commun. Math. Phys. **39**, 131 (1974)
25. Maartens, R., Nel, S.D.: Commun. Math. Phys. **59**, 273 (1978)
26. Coley, A.A.: Gen. Relativ. Gravit. **22**, 3 (1990)
27. Roy, S.R., Prasad, A.: Gen. Relativ. Gravit. **26**, 10 (1994)
28. Bali, R., Jain, S.: Int. J. Mod. Phys. D **16**, 1769 (2007)
29. Bali, R., Meena, B.L.: Pramana **62**, 1007 (2004)
30. Bali, R., Singh, D.K.: Astrophys. Space Sci. **300**, 387 (2005)
31. Pradhan, A., Yadav, V.K.: Int. J. Mod. Phys. D **11**, 893 (2002)
32. Singh, J.K.: Astrophys. Space Sci. **310**, 241 (2007)
33. Borges, H.A., Carneiro, S.: Gen. Relativ. Gravit. **37**, 1385 (2005)
34. Banerjee, N., Das, S.: Gen. Relativ. Gravit. **37**, 1695 (2005)
35. Weinberg, S.: Gravitation and Cosmology. Wiley, New York (1972)
36. Riess, A.G., et al.: Astrophys. J. **607**, 665 (2004)